The comparison of results predicted by the SH and CL theories shows that differences between them increase with the increase of relative thickness, as well as with a rise in the number of an eigenmode (Figs. 1 and 2). For example, for the 0/90/0 lamination with $h_1/h = 0.4$ and $h/a = 0.15$, the frequency ratio associated with the first vibrational mode $\Omega_{00}^{\mathrm{CL}}/\Omega_{00}^{\mathrm{SH}} = 1.16$, whereas for the second mode $\Omega_{01}^{\mathrm{CL}}/\Omega_{01}^{\mathrm{SH}} = 1.56$ (see Fig. 1a, graphs 2 and 4 and graphs 7 and 8).

The convergence of SH and CL results is observed with an increase in Θ (Figs. 2a and 2b).

Acknowledgments

The work of G. Krizhevsky was supported by the Center for Absorption in Science, Ministry of Immigrant Absorption, State of Israel, and the Swope Chair of Mechanics. The work of Y. Stavsky was supported by the Swope Chair of Mechanics and the Technion Fund for Promotion of Research.

References

[1]Johnson, M. W., and Reissner, E., "On Transverse Vibrations of Shallow Spherical Shells," *Quarterly of Applied Mathematics*, Vol. 15, No. 4, 1958, pp. 367–380.

[2]Kalnins, A., "Free Non-Symmetric Vibrations of Shallow Spherical Shells," *Proceedings of 4th U.S. National Congress of Applied Mechanics*, American Society of Mechanical Engineers, New York, 1962, pp. 225–233.

[3]Kalnins, A., "On Vibrations of Shallow Spherical Shells," *Journal of the Acoustical Society of America*, Vol. 33, No. 6, 1961, pp. 1102–1107.

[4]Reissner, E., "Symmetric Bending of Shells of Revolution," *Journal of Mathematics and Mechanics*, Vol. 7, No. 2, 1958, pp. 121–140.

[5]Krizhevsky, G., and Stavsky, Y., "Mindlin-Type Vibration Theory for Laminated Transversally Isotropic Shallow Spherical Shells," *Proceedings of 5th International Conference of Recent Advances in Structural Dynamics* (Southampton, UK), 1994 (to be published).

[6]Soamidas, V., and Ganesan, N., "Asymmetric Vibrations of Layered Shells of Revolution," *Journal of Sound and Vibrations*, Vol. 147, No. 1, 1991, pp. 39–56.

[7]Godunov, S. K., "On the Numerical Solution of Boundary Value Problems for Systems of Linear Ordinary Differential Equations," *Uspekhi Matematicheskikh Nauk (Advances in Mathematical Sciences)*, Vol. 16, No. 3, 1961, pp. 171–174.

Genetic-Algorithm-Based Procedure for Pretest Analysis

Claudio G. Franchi* and Daniele Gallieni*
Politecnico di Milano, Milan 20133, Italy

Nomenclature

$\mathcal{A}$	= $2n_M \times 2n_M$ state matrix
C	= damping $n \times n$ matrix of the system
$\mathcal{F}$	= vector of the $2n_M$ inputs
F	= vector of the n external forces
K	= stiffness $n \times n$ matrix of the system
M	= mass $n \times n$ matrix of the system
q	= vector of the n degrees of freedom
$\mathcal{S}$	= $m \times 2n$ matrix formed by zeros and ones
svd($\mathcal{O}$)	= singular values of the $\mathcal{O}$ matrix
x	= vector of the $2n_M$ states
Δ_j	= integer parameter that assumes the values 0 or 1
ξ_i	= adimensional damping coefficients
ω_i	= natural frequencies of the system
$(\dot{\ })$	= time derivative

Received Jan. 27, 1994; revision received Aug. 23, 1994; accepted for publication Aug. 23, 1994.

*Department of Aerospace Engineering.

I. Introduction

THE optimization of the test setup in experimental modal analysis, the so called pretest analysis, is important to allow the execution of vibration tests on large structures. To reduce measurement time and cost, the test has to be performed using the least number of sensors and shakers that will identify the structural modes in the measured frequency range.[1] This paper issues the optimal layout of sensors for experimental modal testing as a testbed for using genetic-algorithm (GA)[2]-based procedures in the solution of sensors and actuators optimal placement for adaptive structures design.[3–5]

In the pretest phase, only the results of numerical modal analysis are available and used to look for the optimal sensors layout. The sensors position defines the subset of measured degrees of freedom (DOF) in the set of finite element mesh DOFs. Pretest analysis looks for the best choice of the elements of this subset by assembling a reduced experimental model defined by measured DOFs only. In many actual problems the number of DOFs of the finite elements model n is quite large, although the experimental measurements can be performed just at few points. Thus, calling m the number of measured points, $m \ll n$. The problem of finding the best subset of m measurement points in a n sized mesh belongs to the family of combinatorial optimization which requires the simple combinations of n elements in a class of m.

For real problems the number of independent solutions is very large, and enumerative methods become computationally useless. On the other hand, the discrete nature of this problem suggests the avoidance of the use of analytical optimization methods both due to the lack of gradient informations and robustness in direction finding procedures. This kind of problem presents many local optima; and gradient information allows local peaks to be reached leading to solutions depending on the starting conditions. A more feasible approach is represented by a combinatorial method, which treats the coordinate of a grid node as a two-state variable. The latter can be switched on, if chosen to a measure point, or switched off, when inactive. By following this discrete formulation, several heuristic methods have been proposed[6,7] although most of them do not fit completely in the combinatorial nature of the present task. In fact, they cannot treat the whole set of measured DOFs as a single optimization variable but treat each of its elements one at a time. The drawback of finding local optima is overcome by using either simulated annealing or other stochastic algorithms. These methods allow, at least during the initial phase of the optimization, movement toward nonimproving solutions. Moreover, the methods deal with a set of points in solution space, thus avoiding that the solution to be dependent on initial guess.[8]

Many features of the GA meet the discussed requirements and thus, its suitability is suggested for solving these kind of problems.[9] The features are summarized in the following points:

1) GA allows the direct treatment of the set of measured DOFs as a single optimization variable, instead of assigning one variable to any measured DOF. Moreover, a discrete problem can be implemented by considering the integer numbers that represent the DOFs in the finite element model as optimization variables.

2) Like other stochastic methods, GA evaluates at any iteration a number of different solutions equal to the number of individuals of the population. This fact guarantees that the solution will converge to a sufficiently general optimum, regardless of the starting guess. The optimization procedure is based only on the evaluation of the objective function, without any need of gradient information.

3) GA shows all of the generality typical of simulated annealing and other stochastic methods. Its strength comes from the use, beside these techniques, of evolutionary criteria in selection. This fact guarantees a fruitful equilibrium in selecting good solutions among the available ones and exploitation of new directions in solutions space by generating new genetic material from well performing old individuals.

4) The computational effort does not directly depend on the combinatorial order of the problem but is related to the size of the popula-

tion. Finally, GA presents an intrinsic parallelism which permits an easy exploitation of the massive parallelism of modern computers.

II. Problem Formulation

The dynamics of a mechanical system with n degrees of freedom is ruled by the equation

$$M\ddot{q} + C\dot{q} + Kq = F \tag{1}$$

Equation (1) can be written in the n_M modal coordinate η by computing the n_M eigenvalues that lie in the range of frequencies to be measured[10] and the $n \times n_M$ matrix ϕ of the related eigenvectors normalized by

$$\phi^T M \phi = I \tag{2}$$

The second-order equation (1) can be transformed into two first-order equations. The system is thus described in its state space by the equation

$$\dot{x} = \mathcal{A}x + \mathcal{F} \tag{3}$$

and defined by

$$x = (\eta, \dot{\eta})$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathrm{diag}(\omega_i^2) & -\mathrm{diag}(2\xi_i\omega_i) \end{bmatrix}$$

$$\mathcal{F} = (0, \phi^T F)$$

The vector y of the m outputs is obtained from the states by applying an $m \times 2n_M$ output matrix C such as

$$y = Cx \tag{4}$$

The role of the output matrix is to weight the states to select some degrees of freedom or their time derivatives. In fact, Eq. (4) can also be written as

$$Cx = \mathcal{S}\begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} \tag{5}$$

The rank of the observability matrix $\mathcal{O}$, defined by

$$\mathcal{O} = [C \quad C\mathcal{A} \quad C\mathcal{A}^2 \quad \cdots \quad C\mathcal{A}^{2n_M-1}] \tag{6}$$

represents the most immediate index of observability of the system. The system is fully observable if $\mathrm{rank}(\mathcal{O}) = 2n_M$, although it is not if the rank is lower.[10] However, it does not give any information about the level of observability. As an example, a sensor placed near a modal shape node gives a small contribution to $\mathrm{rank}(\mathcal{O})$; it is not zero but lower than any other different place would give. Considering this fact, several authors[11–13] proposed observability indices to measure how much the system is observable. The one used in this paper is defined by

$$I_O = \min[\mathrm{svd}(\mathcal{O})] \tag{7}$$

The use of this index does not reduce the generality of the optimization algorithm presented in this paper because of its independence from the choice of the objective function. On the other hand, neither the observability index here implemented nor the many others that can be used guarantee the correct reconstruction of the modal shapes. In experimental modal analysis, measured DOFs are the only available knowledge on the model. These measurement points must be placed on the actual structure in a convenient layout, allowing the identification of natural frequencies and the associated mode shapes. The measured points are the nodes of the reduced test mesh and must be properly connected in order to reconstruct both the geometry of the structure and the measured mode shapes. Thus, for any measured mode, beside the maximization of the observability, a measure of the goodness in mode shapes reconstruction must

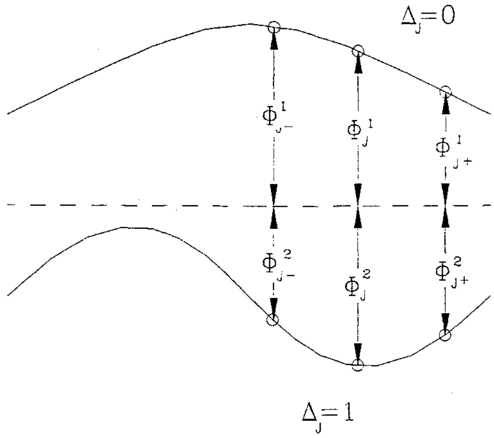

Fig. 1 Evaluation of the reconstruction index.

be evaluated. Hence, in implementing this method a modal reconstruction index is computed. It is a penalty factor I_R evaluated for any individual of the population by

$$I_R = \frac{1}{m}\sum_{j=1}^{m} \Delta_j \tag{8}$$

By denoting as $j+$ and $j-$ the points of the reduced mesh nearest to the jth sensor, Δ_j is 1 if

$$\exists i = 1, \ldots, n_M \quad : \quad \left(\phi_j^i - \phi_{j+}^i\right) \cdot \left(\phi_j^i - \phi_{j-}^i\right) > 0 \tag{9}$$

ϕ_j^i, ϕ_{j+}^i, and ϕ_{j-}^i are the amplitudes of the ith mode at points j, $j+$, and $j-$. Hence, in accordance with Fig. 1, a sensor contributes to the sum in Eq. (8) if a change of slope occurs, at least on one mode shape, between it an its neighbor points. I_R has a value between 0 and 1, which increases as the number of sensors that contribute to Equation (8) increases. Its aim is to avoid the massing of sensors and to foster their scattering on the structure to find the maximal geometrical variety. Thus, starting from the complete finite elements model and by selecting a modal base to be identified, any set of m measured points is evaluated by computing the function

$$f = I_O \cdot I_R \tag{10}$$

III. Genetic Algorithm

A GA is a method in which operators mimicking natural selection guide the random search toward a favorable region in the solution space. GA works over a population of n_P points in the solution space, rather than over a single one, and operates on their binary codification by combining probabilistic and evolutionary rules. Every point in the m dimensional solution space is coded as a binary string made by m substrings, thus, well fitting the problem discussed here. Every variable assumes only integer values, i.e., the number of the DOF that is chosen to be measured. Hence, the number of bits needed to represent an individual of the population is m times the minimum needed to exceed the binary codification of n. GA evaluates and tests a population of individual strings at any iteration, each string describing a point in the m dimensional solutions space. In this paper, the fitness value of the ith individual, which is related to the value of the objective function, is given by

$$\bar{f}_i = \frac{f_i}{\sum_{i=1}^{n_P} f_i} \tag{11}$$

The fitness value of each individual determines the probability of survival in the next generation, i.e., the probability to be selected as parent for chromosomes recombination. The evolutionary procedure to determine the individuals of the new generation is obtained by applying three classical operators in genetic evolution: reproduction, crossover, and mutation presented in Ref. 2. The measure of convergence ε used in this paper is given by

$$\varepsilon = \frac{|f_{\max} - f_{\text{avg}}|}{|f_{\max}|} \tag{12}$$

where $f_{\max}$ is the best individual fitness and f_{avg} is the average one. Convergence is achieved when all of the individuals of a generation are bound to a medium value. The allowed solution may not be the best among all of those evaluated during the evolution. This could seem to be a lack of robustness for this method, but many of the optimization problems suitable for a GA approach have a set of quasioptimal solutions instead of a dominant one. The procedure must guarantee convergence to one of these practical, quasioptimal solutions, and the dispersion of results over a set of feasible good solutions is not a symptom of weakness of the method.

IV. Numerical Examples

To test the proposed method, some numerical examples are analyzed. Two populations of 250 and 500 individuals are tested by considering 100 generations and the crossover and mutation rates are fixed at 0.65 and 0.0015 (Ref. 14).

The pretest analysis of a square, uniform cantilever plate with different boundary conditions has been carried out. Figure 2 presents the first five mode shapes of the cantilever plate. The identification of the modes 1, 2, and 4 by three sensors note the importance of the I_R index. The sensors layout is represented by the symbol • in Fig. 2 and proves the solution converges toward the right layout, even when the last mode observability and reconstruction indices disagree. In fact, major modal participation, hence observability, occurs when the sensors are placed at the unconstrained corners albeit poor information is available to reconstruct the central bump. The symbol o in Fig. 2 represent the sensors layout for the simultaneous identification of whole analyzed mode shapes by five sensors.

Another boundary condition, with two opposite sides clamped, has been analyzed. The identification of the first three mode shapes by three sensors is presented in Fig. 3.

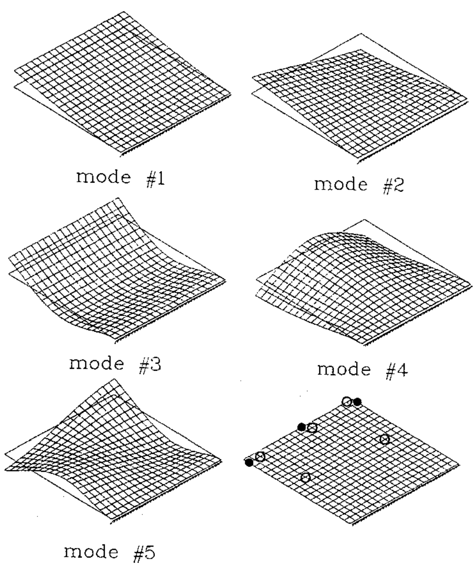

Fig. 2 Cantilever uniform plate, mode shapes and sensors layout.

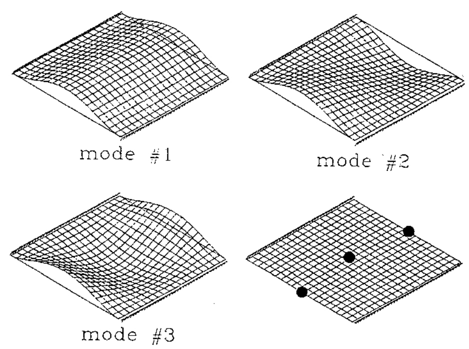

Fig. 3 Uniform plate with two clamped sides, mode shapes and sensors layout.

V. Concluding Remarks

In this paper the features of GA and their major advantages with respect to other analytical, heuristic, or stochastic optimization methods have been pointed out. The algorithm has been applied to solve a problem and has been proved to fulfill its properties: the optimal placement of sensors in pretest analysis. A proper objective function has been defined to include both the requests of observability of the reduced system and of geometrical reconstruction of the measured mode shapes. The results, although obtained from simple examples, confirm all of the premises and are in accordance with physical experience. Moreover, the method here presented seems to be directly suitable for the solution of more general problems concerning the integrated optimization of sensors' and actuators' placement and model reduction. Finally, the codification of the optimization parameters by means of binary strings allows both discrete and continuous variables to be dealt with simultaneously and in a natural way. This fact opens the way to interesting applications in modern integrated optimization, avoiding heavy and unfeasible continuous treatment of discrete variables.

References

[1]Kirkegaard, P. H., and Brincker, R., "On the Optimal Location of Sensors for Parametric Identification of Linear Structural Systems," *17th International Seminar on Modal Analysis*, KU Leuven, 1992, pp. 1177–1190

[2]Goldberg, D. E., *Genetic Algorithms in Search, Optimization and Machine Learning*, Addison–Wesley, Reading, MA, 1989.

[3]Sepulveda, A. E., and Schmit, L. A., Jr., "Optimal Control Placement of Actuators and Sensors in Control Augmented Structural Optimization," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference* (Long Beach, CA), AIAA, Washington, DC, 1990, pp. 217–240.

[4]Ponslet, E., and Haftka, R. T., and Cudney, H., "Optimal Placement of Tuning Masses on Truss Structures by Genetic Algorithm," *Proceedings of the 34th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference* (La Jolla, CA), AIAA, Washington, DC, 1993.

[5]Kammer, D. C., "Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, 1991, pp. 251–259.

[6]Skelton, R. E., and De Lorenzo, M. D., "Space Structure Control Design by Variance Assignment," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 4, 1985, pp. 454–462.

[7]Padula, S. L., and Sandridge, C. A., "Active Strut Placement Using Integer Programming for the CSI Evolutionary Model," *AIAA/USAF/NASA/OAI Symposium on Multidisciplinary Analysis and Optimization* (Cleveland, OH), AIAA, Washington, DC, 1993, pp. 784–793.

[8]Chen, G. S., Bruno, R. J., and Salama, M., "Optimal Placement of Active/Passive Members in Truss Structures Using Simulated Annealing," *AIAA Journal*, Vol. 29, No. 8, 1991, pp. 1327–1334.

[9]Krishnakumar, K., and Goldberg, D. E., "Control System Optimization Using Genetic Algorithms," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 3, 1992, pp. 735–740.

[10]Meirovitch, L., *Dynamics and Control of Structures*, Wiley, New York, 1990.

[11]Hamdan, A. M. A., and Nayfeh, A. H., "Measures of Modal Controllability and Observability for First- and Second-Order Linear Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 3, 1989, pp. 421–428.

[12]Kim, Y., and Junkins, J. L., "A Measure of Controllability for Actuator Placement," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 895–902.

[13]Moore, B. C., "Principal Component Analysis in Linear System: Controllability, Observability and Model Reduction," *IEEE Transaction on Automatic Controls*, Vol. AC-26, No. 1, 1981, pp. 17–32.

[14]De Jong, K., "Adaptive System Design: A Genetic Approach," *IEEE Transactions on Systems Science and Cybernetics*, Vol. SMC-10, No. 9, 1980, pp. 566–574.